Gravitational Radiation Reaction for Inspiralling Binaries
Spin-spin effects to 3.5 post-Newtonian order

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1 Introduction

2 Spin-spin R.R. : equations of motion to 3.5PN order

3 Applications

4 Conclusion and future work
Why study spin effects in binary systems

- Most astrophysical objects are spinning bodies and the orbital motion can be very different from the non-spinning case.
- Spin effects contribute to the gravitational waveform and the overall emission of energy and angular momentum.

- Including spin increases the computational burden and may effect the accuracy on source parameter estimation.
- It’s important to have a complete and reasonable accurate picture of the spin effects in the binary systems.
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### TABLE I. Number of GW inspiral cycles contributed by different PN orders for different NS-BH binaries. We assume $S = 0.3$ and an observation time $T_{\text{obs}} = 1$ yr. In the bottom section of the table, we normalize the number of cycles associated with the Brans-Dicke parameter to $\sigma$ (first row) and to the Cassini bound $\omega_{\text{BD}} > \omega_{\text{Cassini}} = 4 \times 10^4$ (second row). We also show the initial and final GW frequencies, assuming an upper cutoff of 1.0 Hz for the LISA noise curve.

<table>
<thead>
<tr>
<th>PN order</th>
<th>$(1.4 + 400)M_\odot$</th>
<th>$(1.4 + 1000)M_\odot$</th>
<th>$(1.4 + 5000)M_\odot$</th>
<th>$(1.4 + 10^4)M_\odot$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_{\text{in}}$(Hz)</td>
<td>$4.601 \times 10^{-2}$</td>
<td>$3.658 \times 10^{-2}$</td>
<td>$2.446 \times 10^{-2}$</td>
<td>$2.057 \times 10^{-2}$</td>
</tr>
<tr>
<td>$f_{\text{fin}}$(Hz)</td>
<td>1.000</td>
<td>1.000</td>
<td>0.8792</td>
<td>0.4397</td>
</tr>
<tr>
<td>Newtonian</td>
<td>2 294 904</td>
<td>1 828 036</td>
<td>1 224 122</td>
<td>1 025 711</td>
</tr>
<tr>
<td>1PN</td>
<td>35 366</td>
<td>44 712</td>
<td>67 309</td>
<td>78 460</td>
</tr>
<tr>
<td>Tail</td>
<td>$-18 064$</td>
<td>$-29 081$</td>
<td>$-66 278$</td>
<td>$-89 793$</td>
</tr>
<tr>
<td>Spin-orbit</td>
<td>$1437 \beta$</td>
<td>$2314 \beta$</td>
<td>$5274 \beta$</td>
<td>$7145 \beta$</td>
</tr>
<tr>
<td>2PN</td>
<td>422</td>
<td>868</td>
<td>3016</td>
<td>4653</td>
</tr>
<tr>
<td>Spin-spin</td>
<td>$-139 \sigma$</td>
<td>$-288 \sigma$</td>
<td>$-1001 \sigma$</td>
<td>$-1545 \sigma$</td>
</tr>
<tr>
<td>Brans-Dicke</td>
<td>$-3 560 569 \sigma$</td>
<td>$-1 793 782 \sigma$</td>
<td>$-536 954 \sigma$</td>
<td>$-319 126 \sigma$</td>
</tr>
<tr>
<td>Brans-Dicke</td>
<td>$-89 \omega_{\text{Cassini}} / \omega_{\text{BD}}$</td>
<td>$-45 \omega_{\text{Cassini}} / \omega_{\text{BD}}$</td>
<td>$-13 \omega_{\text{Cassini}} / \omega_{\text{BD}}$</td>
<td>$-8.0 \omega_{\text{Cassini}} / \omega_{\text{BD}}$</td>
</tr>
</tbody>
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**PN order counting rules for spinning binaries**

- Post-Newtonian parameter $\epsilon \sim \frac{m}{r} \sim v^2$, $(G = c = 1)$

<table>
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<tr>
<th>For arbitrary rotating objects</th>
<th>Rapidly rotating compact objects</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{v} \sim O(v)$, $\bar{x} \sim O(r)$</td>
<td>$\bar{v} \sim O(1)$, $\bar{x} \sim O(m)$</td>
</tr>
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<td>$\Rightarrow S_A \sim O(mvr) \sim O(\epsilon^{3/2})r^2$</td>
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<tr>
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<td>$a_{SO} \sim \frac{1}{m^2} S \sim \frac{m}{r} v^2 \rightarrow 1PN$</td>
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<td>$a_{SS} \sim \frac{1}{mr^2} S^2 \sim \frac{m}{r^2} v^2 \rightarrow 1PN$</td>
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- For our calculation, we assume arbitrary rotating objects.
Post-Newtonian parameter $\epsilon \sim \frac{m}{r} \sim v^2$, ($G = c = 1$)

For arbitrary rotating objects

- $v \sim O(v)$, $\vec{x} \sim O(r)$
- $S_A \sim O(mvr) \sim O(\epsilon^{3/2})r^2$
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- Leading order equation of motion with spin:
  - $a_{SO} \sim \frac{v}{r^3}S \sim \frac{m}{r^2}v^2 \rightarrow 1PN$
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For rapidly rotating compact objects

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Leading order spin-spin R.R. is 3.5PN

- Equation of motion in post-Newtonian language:
  \[ a = a_N(PM) + a_{1PN}(PM, SO, SS) + a_{2PN}(PM, SO, SS) + a_{2.5PN}(PM) 
  + a_{3PN}(PM, SO, SS) + a_{3.5PN}(PM, SO, SS) + \ldots \]

- No radiation reaction in N, 1PN, 2PN, 3PN EOM.
- No spin in quadrupole formula \( \Rightarrow \) No spin in 2.5PN order R.R.
- Leading order R.R. containing spin: 3.5PN order
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Calculate the spin-spin effects

- Pati & Will (2002) worked out the EOM up to 3.5PN order in terms of $I^{ij\cdots}$, $J^{ij\cdots}$, $U(x)$, $X(x)$, \ldots.
- Assume the objects are perfect fluid balls.
- Define the ordinary and proper spin:
  \[
  S^i_A = \epsilon^{ijk} \int_A \rho^* \bar{x}^j \bar{v}^k d^3x,
  \]
  where $\bar{v}^i = v^i - v'_A$, $\bar{x}^i = x^i - x'_A$
  \[
  S^i_A \equiv S^i_A (1 + \frac{1}{2} v_A^2 + 3 \frac{m_B}{r}) - \frac{1}{2} [v_A \times (v_A \times S_A)]^i - S^i_A \mathcal{T}^{ij} + S^i_A \mathcal{T}^{ij}.
  \]
  Using expansion parameters $\bar{x}, \bar{x}', \bar{v}, \bar{v}'$, we expand EOM w.r.t. center of each bodies, only keep terms proportional to $(\rho^* \bar{x}\bar{v})(\rho^* \bar{x}'\bar{v}')$.
- Spin-spin R.R. (3.5PN SS EOM) comes from (1) $a^{i}_{3.5PN}$, (2) $a^{i}_{1PN} + a^{i}_{2.5PN}$, (3) 2.5PN part in $S^i$.
- We can also calculate the equations of spin to this order as
  \[
  \dot{S}^i_A = \epsilon^{ijk} \int_A \rho^* \bar{x}^j a^k_{3.5PN} d^3x
  \]
The results

\[ a_{3.5PN-SS}^i = \frac{1}{30r^n} \left\{ -1229m (v \cdot \mathbf{S}_2) + 1317mr (n \cdot \mathbf{S}_2) - 1125r^2 (v \cdot \mathbf{S}_2)r \\
+ 81 (v \cdot \mathbf{S}_2) r^2 + 1155r^3 (n \cdot \mathbf{S}_2) r + 225r (n \cdot \mathbf{S}_2) r^2 \right\} S_i^i \\
+ \left[ -1229m (v \cdot \mathbf{S}_1) + 1317mr (n \cdot \mathbf{S}_1) - 1125r^2 (v \cdot \mathbf{S}_1)r \\
+ 81 (v \cdot \mathbf{S}_1) r^2 + 1155r^3 (n \cdot \mathbf{S}_1) r + 225r (n \cdot \mathbf{S}_1) r^2 \right\} S_i^i \\
+ \left[ -570m (S_1 \cdot S_2) + 4788m (n \cdot S_1)(n \cdot S_2) - 5850 (S_1 \cdot S_2)r^2r \\
+ 1026 (S_1 \cdot S_2) r^3r + 2880 (v \cdot S_1) (S_1 \cdot S_2)r \\
+ 41580r^2 (n \cdot S_1)(n \cdot S_2)r - 13140 (n \cdot S_1)(v \cdot S_2)r \\
- 13140 (v \cdot S_1)(n \cdot S_2)r - 5220 (n \cdot S_1)(n \cdot S_2)r^2 \right\} v_i^i \\
+ \left[ 1809m (S_1 \cdot S_2) r - 11766m (n \cdot S_1)(n \cdot S_2)r \\
+ 2607m (n \cdot S_1)(v \cdot S_2) + 2607m (n \cdot S_1)(v \cdot S_2) \\
+ 8510 (S_1 \cdot S_2) r^3r - 2970 (S_1 \cdot S_2) v^3r \\
- 17980 (n \cdot S_1)(n \cdot S_2)r^3r + 30870 (n \cdot S_1)(v \cdot S_2)r^3r \\
+ 30870 (v \cdot S_1)(n \cdot S_2) N v^2r - 10080 (v \cdot S_1)(v \cdot S_2)r \\
+ 21420 (n \cdot S_1)(n \cdot S_2)r v^2r - 3840 (v \cdot S_2)(n \cdot S_1)r v^2 \\
- 3590 (v \cdot S_1)(n \cdot S_2)r v^2 \\
- 720m [(n \times S_1) \cdot v](n \times S_2)v - 720m [(n \times S_2) \cdot v](n \times S_1)v \right\} \]

\[ S_i^i = \frac{\mu}{r^4} \left\{ (v \times S_1)^i \left[ \frac{61}{5} \frac{n \cdot S_2}{r} + 3r^2 (n \cdot S_2) - \frac{3}{5} (n \cdot S_2)v^2 \right] \\
+ \frac{148}{15} m (v \cdot S_2) - \frac{33}{r} m (n \cdot S_2) - 9 (n \cdot S_2)v^2 \right\} \\
+ 21 (n \cdot S_2)r^3 - 12r^2 (v \cdot S_2) + \frac{12}{5} v^2 (v \cdot S_2) \]

- The potential form (Pati & Will 2002)
- SO terms (Will 2005)
- SS terms (Wang & Will 2006)
Near zone orbital energy and total angular momentum loss due to the leading order spin-spin gravitational radiation:

\[
\dot{E}_N^{(SS)} = \mu (\mathbf{v} \cdot \mathbf{a}_{3.5\text{PN} - SS}) \\
\dot{J}_N^{(SS)} = \dot{L}_N^{(SS)} + \dot{S}_N^{(SS)} \\
\dot{L}_N^{(SS)} = \mu (\mathbf{x} \times \mathbf{a}_{3.5\text{PN} - SS})
\]

Comparing \(\dot{E}_N^{(SS)}\) and \(\dot{J}_N^{(SS)}\), with the radiation zone energy (angular momentum) flux (Kidder 1995), we can show the difference is a total time derivative.

\[\Rightarrow \text{Pure gauge effect, re-define the near zone energy(angular momentum).}\]

- Near zone energy (angular momentum) loss and radiation zone flux balance. ✓
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Conclusion

- We calculated the 3.5 post-Newtonian order spin-spin radiation reaction and equation of spin for inspiralling binary systems.
- Using the our result, we calculated the near zone energy and angular momentum loss, which agrees with the radiation zone flux formula.
- Because of the spin-spin coupling, \( \dot{\mathbf{S}} \) in this order is no longer a total time derivative and it has contribution to the spin precession.

Possible future works

- Detail investigation of the gauge freedom in the near zone.
- Spin-spin contribution to the 2PN equation of motion.
- High order spin-orbit and spin-spin contribution to the waveform.
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