Mode-sum regularization of the scalar self-force in Schwarzschild spacetime

Roland Haas and Eric Poisson
Department of Physics, University of Guelph

To formulate the equations of motion of a small body in a specified background spacetime, beyond the test-mass approximation. This first step was made back in 1997 [Mino, Tanaka, and Sasaki (1997); Quinn and Wald (1997)].

To concretely describe this motion for situations of astrophysical interest (generic orbits of a Kerr black hole). Much recent progress on this front, mostly for Schwarzschild spacetime.

To properly incorporate this information into a wave-generation formalism. The holy grail: still elusive.
A particle with scalar charge $q$ moves in spacetime.

It generates a field $\Phi$ that satisfies a wave equation with a distributional source term.

The field is singular on the world line, but it is regularized with the **Detweiler-Whiting prescription** \[\text{[Detweiler and Whiting (2003)]}\]

\[
\Phi^{\text{ret}} = \Phi^{S} + \Phi^{R}
\]

where the singular field is known locally.

The scalar self-force on the particle is produced by the regular field \[\text{[Quinn (2000)]}\]

\[
ma^{a} = q(g^{ab} + u^{a}u^{b})\nabla_{b}\Phi^{R}
\]
In Schwarzschild spacetime, the retarded field must be calculated numerically via a mode decomposition

$$\Phi^{\text{ret}} = \sum_{lm} \Phi_{lm}(t, r) Y_{lm}(\theta, \phi)$$

Each mode satisfies a reduced wave equation

$$\left\{ -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^*} - f \left[ \frac{l(l+1)}{r^2} + \frac{2M}{r^3} \right] \right\} \left( r \Phi_{lm} \right) = -\frac{4\pi q f}{r u t} Y_{lm}(\frac{\pi}{2}, 0) e^{-im\varphi(t)} \delta \left( r - r(t) \right)$$

$$f = 1 - \frac{2M}{r} \quad r^* = r + 2M \ln \left( \frac{r}{2M} - 1 \right) \quad r(t), \varphi(t) : \text{particle's world line}$$

Each mode $\Phi_{lm}(t, r)$ is bounded at $r = r(t)$. 
The singular field is known analytically as an expansion in powers of $s :=$ distance from world line,

$$\nabla_a \Phi^S = g_a^\mu \left[ \frac{1}{s^2} M^\mu_a + \frac{1}{s} N^\mu_a + O^\mu_a + s P^\mu_a + \cdots \right]$$

where $M^\mu_a$, $N^\mu_a$, $\ldots$ involve tensors and bitensors evaluated on the world line.

This vector is converted into four scalars by an orthonormal tetrad $e_a^\mu$, and each scalar is decomposed in spherical harmonics.

This requires an arbitrary, but smooth extension of the local expansion over the whole sphere.

[For example: $\theta - \frac{\pi}{2} = -\cos(\theta) - \frac{1}{6} \cos^3(\theta) + \cdots$]
This yields

\[ \left( e_{(\mu)}^a \nabla_a \Phi^S \right)_l = (l + \frac{1}{2}) A(\mu) + B(\mu) + \frac{C(\mu)}{(l + \frac{1}{2})} + \frac{D(\mu)}{(l - \frac{1}{2})(l + \frac{3}{2})} + \cdots \]

The regularization parameters \( A(\mu), B(\mu), \ldots \) are independent of \( l \) but depend on \( r(t) \) and \( \varphi(t) \).

Removal of this from \( \left( e_{(\mu)}^a \nabla_a \Phi^{\text{ret}} \right)_l \) leads to a converging mode-sum for the (tetrad projections of the) self-force.
Some of this work is repetition of older work [Barack and Ori (2002); Mino, Nakano, and Sasaki (2002); Detweiler, Messaritaki, and Whiting (2003); Kim (2004)].

The novel aspects are

- the involvement of a tetrad, to expand scalars (not vector components) in spherical harmonics
- the first calculation of $D(\mu)$ for general orbits in Schwarzschild spacetime
- the first complete computation of the scalar self-force for noncircular orbits in Schwarzschild (Roland Haas)
Self-force for circular orbit \((r = 6M)\)