A Higher Dimensional Stationary Rotating Black Hole Must be Axisymmetric
— Black Hole Rigidity in Higher Dimensions —

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Outline

- Introduction/Motivation
- Rigidity Theorem
- Key Issues & Sketch of Proof
- Remarks
Why Higher Dimensions?

- required in most attempts to unify the forces in Nature
  Kalzu-Klein, Supergravity, Superstring theories

- Braneworld / Large extra-dimensions phenomenology
  \( D > 4 \) BHs and Hawking radiation at LHC

- help understand 4-dimensional gravity
  HD BH solutions play an important role

Focus:
Stationary Black Holes in \( D > 4 \) General Relativity
- with no compactified dimensions
Asymptotically flat, Stationary 4\(D\) BHs

- **Exact Solutions** — (Kerr metric)
- **Stability** — (Stable \(\Rightarrow\) final state of collapse)
- **Topology** — (Event Horizon \(\approx 2\)-sphere \(\times\) \(\mathbb{R}\))
- **Symmetry** — (static, or axisymmetric)
- **Uniqueness** — (Vacuum \(\Rightarrow\) Kerr-family)
- **BH Mechanics** — (Thermodynamics)

**Which properties of 4\(D\) BHs are extended to \(D > 4\)?**
Recent Results of $D > 4$ BHs

- **Exact Solutions** — more variety
  Rotating Holes [Myers & Perry], Rotating Rings [Emparan & Reall]

- **Stability** — partial results
  Static vacuum $\Rightarrow$ stable, [Al & Kodama], Rotating holes $\Rightarrow$ not fully studied yet

- **Topology** — more variety
  Some restrictions, e.g., [Galloway & Shoen]

- **Symmetry / “Rigidity”** — This talk

- **Uniqueness** — non-unique
  holes and ring w/ the same $(J, M)$
Black Hole “Rigidity”

Let \((M, g)\) be a \(D \geq 4\), analytic, asymptotically flat, stationary vacuum solution to Einstein’s equation. Assume event horizon \(\mathcal{H}\) is analytic, non-degenerate, and topologically \(\mathbb{R} \times \Sigma\) with \(\Sigma\) being compact, connected.

**Theorem 1:** There exists a Killing field \(K^a\) in entire D.O.C. such that \(K^a\) is normal to \(\mathcal{H}\) and commutes with the stationary Killing vector filed \(t^a\) \(\Rightarrow\) “Killing horizon”

**Theorem 2:** If \(t^a\) is not normal to \(\mathcal{H}\), i.e., \(t^a \neq K^a\), then there exist mutually commuting Killing vector fields \(\varphi^a_{(1)}, \cdots, \varphi^a_{(j)}\) \((j \geq 1)\) with period \(2\pi\) and \(t^a = K^a + \Omega_{(1)}\varphi^a_{(1)} + \cdots + \Omega_{(j)}\varphi^a_{(j)}\), where \(\Omega_{(j)}\)’s constants. \(\Rightarrow\) “Axisymmetry” (“Rigid-rotation”)
Why “Rigidity” interesting?

- relates “global” (even horizon) to “local” (Killing horizon)
- rotating hole ⇒ extra-(axial) symmetry
- foundation of BH Thermodynamics
  (Constancy of surface gravity ⇒ Oth Law)
- a necessary step toward “Uniqueness” in $4D$ case

However, Hawking’s proof (1972) for $4D$ case relies heavily on the fact that event horizon cross-section $\Sigma$ is topologically 2-sphere ⇒ Generalization to $D > 4$ is highly non-trivial

Goal: Present a proof of BH Rigidity Theorem in $D \geq 4$ with No Assumption on Topology of Event Horizon
Brief Sketch of Proof of Theorem 1

Step 1
Construct a “candidate” Killing field $K^a$ on $\mathcal{H}$ which satisfies

- $K^a K_a = 0$ and $\mathcal{L}_t K^a = 0$ on $\mathcal{H}$,
- $\alpha = \text{const.} \ (K^c \nabla_c K^a = \alpha K^a)$ on $\mathcal{H}$,
- $\mathcal{L}_K g_{ab} = 0$ on $\mathcal{H}$,

Try this one! $K^a = t^a - s^a$

Step 2
- Show Taylor expansion, 
  $\partial^m (\mathcal{L}_K g_{ab}) / \partial \lambda^m = 0$, at $\mathcal{H}$
- Extend $K^a$ to the whole spacetime by invoking analyticity.
However, there is **No reason why** \( \alpha \) **need be constant.**

— wish to find “correct” \( \tilde{K}^a \) with \( \tilde{\alpha} = \text{const.} =: \kappa \) on \( \mathcal{H} \) by choosing a new, “correct” foliation \( \tilde{\Sigma} \).

Both \( K^a \) and \( \tilde{K}^a \) are null,

\[
\tilde{K}^a = f(x) \ K^a
\]

**Task:** Find a solution to Equation for coordinate transformation from trial \( \Sigma \) to correct \( \tilde{\Sigma} \):

\[
-\mathcal{L}_s f(x) + \alpha(x) \ f(x) = \kappa
\]

\[
K^a + s^a = t^a = \tilde{K}^a + \tilde{s}^a
\]
Find correct foliation $\tilde{\Sigma}$ 4D case

In 4D, horizon cross-section $\Sigma$ is 2-sphere, and therefore the orbits of $s^a$ must be closed.

There is a discrete isometry “$\Gamma$” which maps each null generator into itself.

Discrete isometry, $\Gamma$, helps

- to define the surface gravity as
  \[ \kappa \equiv P^{-1} \int_0^P \alpha[\phi_s(x)] \, ds, \]
- to find correct foliation $\tilde{\Sigma}$
- to show Step 2
Find correct foliation $\tilde{\Sigma} \quad D > 4$ case:

No reason that the isometry $s^a$ need have closed orbits on $\Sigma$.  
$\Rightarrow$ in general, there is No discrete isometry $\Gamma$.

e.g., 5D Myers-Perry BH w/ 2-rotations $\Omega(1), \Omega(2)$:

$\Sigma \approx S^3, \quad t^a = K^a + s^a$

$s^a = \Omega(1)\varphi^a(1) + \Omega(2)\varphi^a(2)$

Each rotation Killing vector $\varphi^a$ has closed orbits but $s^a$ does not if $\Omega(1)$ and $\Omega(2)$ are incommensurable.
Solution to $D > 4$ case:

(i) When $s^a$ has closed orbits on $\Sigma \Rightarrow$ we are done!

\[
\kappa = \frac{1}{P} \int_0^P \alpha[\phi_s(x)] ds \quad P: \text{period} \quad \phi_s: \text{isom. on } \Sigma \text{ by } s^a
\]

(ii) When $s^a$ has No closed orbits $\Rightarrow$ Use Ergodic Theorem!

\[
\kappa = \lim_{T \to \infty} \frac{1}{T} \int_0^T \alpha[\phi_s(x)] ds = \frac{1}{\text{Area}(\Sigma)} \int_{\Sigma} \alpha(x) d\Sigma
\]

“time-average” “space-average”

- can show that the limit “$\kappa$” exists and is constant
- can find well-behaved transformation $\Sigma \to \tilde{\Sigma}$
Remarks

— can apply to any “horizon” defined as the “boundary” of causal past of a complete timelike orbits of $t^a$, e.g., cosmological constant.

— apply to Einstein-$\Lambda$-Maxwell system

— can remove analyticity assumption for the BH interior

When $t^a$ is normal to $H$ ($t^a = K^a$), ⇒ black hole is static.